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The polar analysis of a third order piezoelectricity-like plane tensor

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Abstract

The derivation of the invariants for a tensor of the third order in a two dimensional space is presented in the paper. The mathematical technique to obtain the expression of the invariants is based upon the so-called polar method, proposed as early as 1979 by G. Verchery as an alternative method to represent a plane tensor by its invariants. The polar components of a third order planar tensor are also introduced. When considered in the framework of piezoelectricity, an energetic interpretation of these components is given.

Keywords: Invariants; Third order tensors; Piezoelectricity; Polar method

1. Introduction

Tensors of the third order appear in some fields of interest, for instance in the case of piezoelectricity, where the Cauchy's stress tensor $\boldsymbol{\sigma}$ and the electric displacement $\mathbf{D}$ are linked to the strain tensor $\mathbf{S}$ and to the electric field vector $\mathbf{E}$ by the relation (see for instance Milton, 2002)

$$\begin{aligned} \sigma &= \mathbb{C}\mathbf{S} - \mathbf{e}\mathbf{E} \rightarrow \sigma_{ij} = \mathbb{C}_{ijkl}S_{kl} - e_{kij}E_k, \\ \mathbf{D} &= \mathbf{e}\mathbf{S} + \boldsymbol{\varepsilon}\mathbf{E} \rightarrow D_i = e_{ijk}S_{jk} + \varepsilon_{ij}E_j, \end{aligned} \tag{1}$$

where $\mathbb{C}$ is the fourth order elasticity tensor at null electric field, $\mathbf{e}$ the third order tensor of piezoelectricity coefficients and ε the second order dielectric permittivity tensor at null strain field.

As a matter of fact, the knowledge of tensor invariants can be rather important in several circumstances. In a sense, they provide overall information about the behaviour of the linear application. For a physical problem described by tensors, each tensor has a precise meaning and its invariants can be useful and rather

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meaningful quantities. Just as an easy example, the case of a second order symmetric tensor **L** in a two dimensional space can be cited: its determinant can be expressed as the difference of two independent invariants

$$\det \mathbf{L} = T^2 - R^2, \tag{2}$$

where T and R are respectively the abscissa of the centre of the Mohr's circle and its radius. It is well known that T is linked to the spherical part of **L** and R to its deviatoric part and the importance of this tensor decomposition is well known in mechanics.

Another example, not so well known but rather instructive for the purpose of the present paper, concerns a tensor **T** of the type of elasticity in a bidimensional space: in this case Verchery found in 1979 a complete set of independent invariants that he called the *polar components*. These are linked to the Cartesian components of **T** by the following relations involving complex quantities:

$$\begin{aligned}
8T_0 &= T_{xxxx} - 2T_{xxyy} + 4T_{xyxy} + T_{yyyy},\\
8T_1 &= T_{xxxx} + 2T_{xxyy} + T_{yyyy},\\
8R_0 e^{4i\Phi_0} &= T_{xxxx} - 2T_{xxyy} - 4T_{xyxy} + T_{yyyy} + 4i(T_{xxxy} - T_{xyyy}),\\
8R_1 e^{2i\Phi_1} &= T_{xxxx} - T_{yyyy} + 2i(T_{xxxy} + T_{xyyy}).
\end{aligned} \tag{3}$$

In the case of the elasticity tensor, the four parameters T_0, T_1, R_0 and R_1 have the dimensions of an elastic modulus and are invariants; the fifth invariant is the difference $\Phi_0 - \Phi_1$ of the two polar angles. The reversals of Eqs. (3) are

$$\begin{aligned}
T_{xxxx} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 + 4R_1\cos 2\Phi_1,\\
T_{xxxy} &= R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1,\\
T_{xxyy} &= -T_0 + 2T_1 - R_0\cos 4\Phi_0,\\
T_{xyxy} &= T_0 - R_0\cos 4\Phi_0,\\
T_{xyyy} &= -R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1,\\
T_{yyyy} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 - 4R_1\cos 2\Phi_1.
\end{aligned} \tag{4}$$

It can be shown that this tensor representation has some features (Verchery, 1979; Vannucci, 2002): first of all, the polar invariants have a physical meaning concerning the elastic symmetries:

- a layer is orthotropic if and only if $\Phi_0 - \Phi_1 = K\frac{\pi}{4}, K \in \mathbb{N}$;
- a layer is R_0-orthotropic if and only if $R_0 = 0$;
- a layer is square-symmetric orthotropic if and only if $R_1 = 0$;
- a layer is isotropic if and only if $R_0 = R_1 = 0$.

Secondly, the strain energy can be written in the form

$$W = W_S + W_D, \tag{5}$$

where

$$\begin{aligned}
W_S &= Tt,\\
W_D &= R\ r\cos 2(\Phi - \varphi),
\end{aligned} \tag{6}$$

are respectively the part of W depending on the spherical and deviatoric parts of $\boldsymbol{\sigma}$ and Φ. T, R, Σ are the polar components of $\boldsymbol{\sigma}$ and t, r and φ those of **S** (see for instance Vannucci, 2002, 2005 for more details). When σ is expressed by Eqs. (1), in the absence of **E** we get

$$\begin{aligned}
W_S &= 4T_1t^2 + 4R_1rt\cos 2(\Phi_1 - \varphi),\\
W_D &= 2T_0r^2 + 2R_0r^2\cos 4(\Phi_0 - \varphi) + 4R_1rt\cos 2(\Phi_1 - \varphi).
\end{aligned} \tag{7}$$

Eqs. (7) show that T_1 concerns only the spherical part W_S, that T_0, R_0 and Φ_0 concern only the deviatoric part W_D and that R_1 and Φ_1 couple the two parts W_S and W_D, so that the decomposition of W into spherical and deviatoric parts is possible only for plies that are at least square symmetric (like for instance the case of composite plies where the matrix is reinforced by balanced fabrics).

In this paper, the method proposed by Verchery to find a third-order tensor invariants in a two dimensional space is presented and the tensor invariants found. The polar parameters of such a tensor are also introduced and, for the case of the piezoelectricity tensor, their energetic meaning discussed. The reader is addressed to the original paper by Verchery, 1979, for an introduction to the polar method, or also to (Vannucci and Verchery, 2001), or to (Vannucci, 2001, 2002, 2005), for a detailed presentation of the method.

2. The Verchery's transformation for a third order tensor

The transformation of Verchery is a complex variable map, in a sense analogous to the transformation of Green and Zerna, 1954, interpreted as a complex change of frame: it gives the contravariant components $\mathbf{X}^{\text{cont}} = (X^1, X^2)$ of a vector $\mathbf{x} = (x, y)$ as

$$\begin{aligned} X^1 &= \frac{1}{\sqrt{2}}\bar{k}z = \frac{x+y-\mathrm{i}(x-y)}{2}, \\ X^2 &= \frac{1}{\sqrt{2}}k\bar{z} = \frac{x+y+\mathrm{i}(x-y)}{2} = \bar{X}^1, \end{aligned} \tag{8}$$

where z is the complex variable, $z = x + \mathrm{i}\, y$, and $k = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$, while a bar over the letter indicates the complex conjugate. It is then easy to check that

$$\mathbf{X}^{\text{cont}} = \mathbf{m}_1 \mathbf{x}, \tag{9}$$

with the transformation matrix $\mathbf{m}_1$ given by

$$\mathbf{m}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \bar{k} & k \\ k & \bar{k} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1-\mathrm{i} & 1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{bmatrix}. \tag{10}$$

For the sake of completeness, the covariance components are also introduced hereafter, though not strictly necessary for the developments. The covariant components $\mathbf{X}_{\text{cov}} = (X_1, X_2)$ can be obtained with the use of the metric tensor $\mathbf{g}$, expressing the length $\mathrm{d}s$ of an infinitesimal arc:

$$\mathrm{d}s^2 = \mathrm{d}z\,\mathrm{d}\bar{z} = 2\mathrm{d}X^1\,\mathrm{d}X^2 = g_{ij}\mathrm{d}X^i\,\mathrm{d}X^j, \tag{11}$$

which gives the metric tensor

$$\mathbf{g}_{\text{cov}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

so that one gets

$$\mathbf{X}_{\text{cov}} = \mathbf{g}_{\text{cov}} \mathbf{X}^{\text{cont}} \quad \Rightarrow \quad \begin{cases} X_1 = X^2 = \frac{1}{\sqrt{2}}k\bar{z}, \\ X_2 = X^1 = \frac{1}{\sqrt{2}}\bar{k}z. \end{cases} \tag{13}$$

It is easily recognized, by Eqs. (13), that to obtain the covariant components it suffices, in Eq. (9), to replace the matrix $\mathbf{m}_1$ by its inverse.

An important feature is that the properties of the matrix $\mathbf{m}_1$, which operates the transformation for vectors, i.e. for rank-1 tensors, belong also to the matrices $\mathbf{m}_j$ which operate the transformation for higher order, rank j, tensors. The following formulæ summarize these properties:

$$
\begin{aligned}
&\mathbf{m}_j = \mathbf{m}_j^T,\\
&\mathbf{m}_j^*(=\bar{\mathbf{m}}_j^T) = \bar{\mathbf{m}}_j,\\
&\mathbf{m}_j^*\mathbf{m}_j = \mathbf{I} \quad \Rightarrow \quad \mathbf{m}_j^{-1} = \mathbf{m}_j^*,\\
&\mathbf{m}_j^{-1} = \bar{\mathbf{m}}_j,\\
&\mathbf{m}_j \neq \mathbf{m}_j^*.
\end{aligned}
\tag{14}
$$

These properties characterize the transformation of Verchery: unlike the transformation of Green and Zerna, that of Verchery is unitary, symmetric with respect to both the diagonals, though not Hermitian. An important point is that to get the inverse of the transformation matrix it suffices to substitute imaginary unit i with $-$i.

The transformation matrices for higher order tensors can easily be obtained by a recursive scheme:

$$
\mathbf{m}_j = \begin{bmatrix} m_1^{11}\mathbf{m}_{j-1} & m_1^{12}\mathbf{m}_{j-1} \\ m_1^{21}\mathbf{m}_{j-1} & m_1^{22}\mathbf{m}_{j-1} \end{bmatrix}. \tag{15}
$$

Hence, for a third order tensor, some simple calculations give the double symmetric matrix

$$
\mathbf{m}_3 = \begin{bmatrix} m_1^{11}\mathbf{m}_2 & m_1^{12}\mathbf{m}_2 \\ m_1^{21}\mathbf{m}_2 & m_1^{22}\mathbf{m}_2 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix}
-k & \bar{k} & \bar{k} & k & \bar{k} & k & k & -\bar{k} \\
\bar{k} & -k & k & \bar{k} & k & \bar{k} & -\bar{k} & k \\
\bar{k} & k & -k & \bar{k} & k & -\bar{k} & \bar{k} & k \\
k & \bar{k} & \bar{k} & -k & -\bar{k} & k & k & \bar{k} \\
\bar{k} & k & k & -\bar{k} & -k & \bar{k} & \bar{k} & k \\
k & \bar{k} & -\bar{k} & k & \bar{k} & -k & k & \bar{k} \\
k & -\bar{k} & \bar{k} & k & \bar{k} & k & -k & \bar{k} \\
-\bar{k} & k & k & \bar{k} & k & \bar{k} & \bar{k} & -k
\end{bmatrix}. \tag{16}
$$

So, for a third order plane tensor whose Cartesian components are ranged in the column vector

$$
\mathbf{T} = \begin{Bmatrix} T_{xxx} \\ T_{xxy} \\ T_{xyx} \\ T_{xyy} \\ T_{yxx} \\ T_{yxy} \\ T_{yyx} \\ T_{yyy} \end{Bmatrix}, \tag{17}
$$

it is, like in Eq. (9),

$$
\mathbf{T}^{\text{cont}} = \mathbf{m}_3\mathbf{T}, \tag{18}
$$

and hence

$$\begin{aligned}
T^{111} &= \frac{1}{4}[(1+\mathrm{i})(-T_{xxx}+T_{xyy}+T_{yxy}+T_{yyx})+(1-\mathrm{i})(T_{xxy}+T_{xyx}+T_{yxx}-T_{yyy})],\\
T^{112} &= \frac{1}{4}[(1+\mathrm{i})(-T_{xxy}+T_{xyx}+T_{yxx}+T_{yyy})+(1-\mathrm{i})(T_{xxx}+T_{xyy}+T_{yxy}-T_{yyx})],\\
T^{121} &= \frac{1}{4}[(1+\mathrm{i})(T_{xxy}-T_{xyx}+T_{yxx}+T_{yyy})+(1-\mathrm{i})(T_{xxx}+T_{xyy}-T_{yxy}+T_{yyx})],\\
T^{122} &= \frac{1}{4}[(1+\mathrm{i})(T_{xxx}-T_{xyy}+T_{yxy}+T_{yyx})+(1-\mathrm{i})(T_{xxy}+T_{xyx}-T_{yxx}+T_{yyy})],\\
T^{211} &= \frac{1}{4}[(1+\mathrm{i})(T_{xxy}+T_{xyx}-T_{yxx}+T_{yyy})+(1-\mathrm{i})(T_{xxx}-T_{xyy}+T_{yxy}+T_{yyx})],\\
T^{212} &= \frac{1}{4}[(1+\mathrm{i})(T_{xxx}+T_{xyy}-T_{yxy}+T_{yyx})+(1-\mathrm{i})(T_{xxy}-T_{xyx}+T_{yxx}+T_{yyy})],\\
T^{221} &= \frac{1}{4}[(1+\mathrm{i})(T_{xxx}+T_{xyy}+T_{yxy}-T_{yyx})+(1-\mathrm{i})(-T_{xxy}+T_{xyx}+T_{yxx}+T_{yyy})],\\
T^{222} &= \frac{1}{4}[(1+\mathrm{i})(T_{xxy}+T_{xyx}+T_{yxx}-T_{yyy})+(1-\mathrm{i})(-T_{xxx}+T_{xyy}+T_{yxy}+T_{yyx})].
\end{aligned} \tag{19}$$

In Eqs. (19), it can be noticed that all the components are complex quantities, and that

$$T^{222}=\bar{T}^{111},\quad T^{221}=\bar{T}^{112},\quad T^{212}=\bar{T}^{121},\quad T^{211}=\bar{T}^{122}, \tag{20}$$

so only four complex contravariant components of $\mathbf{T}$ must be known, in place of eight real Cartesian components. For what concerns the covariant components, according to the general properties of the transformation matrices given in Eqs. (14) and as a consequence of the fact that $\mathbf{T}$ is real, they are the complex conjugates of the contravariant ones:

$$\mathbf{T}_{\mathrm{cov}}=\mathbf{m}_3^{-1}\mathbf{T}=\bar{\mathbf{m}}_3^{\mathbf{T}} \quad \Rightarrow \quad T_{ijk}=\bar{T}^{ijk} \quad \forall i,j,k=1,2. \tag{21}$$

The first equality in Eq. (21) can be easily found by using Eqs. (12) and considering that, generally speaking, $T_{mnp}=T^{ijk}g_{mi}g_{nj}g_{pk}$.

3. Third order tensors with the symmetries of piezoelectricity

Let us consider now a tensor for which, like in the case of piezoelectricity, the Cartesian components are subjected to the following symmetry prescriptions:

$$T_{ijk}=T_{ikj}. \tag{22}$$

For a plane tensor, Eq. (22) reduce to the two conditions

$$T_{xxy}=T_{xyx},\quad T_{yxy}=T_{yyx}, \tag{23}$$

so that for plane tensors satisfying Eqs. (22) the independent components are only six: the contravariant components in Eqs. (19) become in this case

$$\begin{aligned}
T^{111} &= \frac{1}{4}[(1+\mathrm{i})(-T_{xxx}+T_{xyy}+2T_{yxy})+(1-\mathrm{i})(2T_{xxy}+T_{yxx}-T_{yyy})],\\
T^{112} &= \frac{1}{4}[(1+\mathrm{i})(T_{yxx}+T_{yyy})+(1-\mathrm{i})(T_{xxx}+T_{xyy})]=T^{121},\\
T^{122} &= \frac{1}{4}[(1+\mathrm{i})(T_{xxx}-T_{xyy}+2T_{yxy})+(1-\mathrm{i})(2T_{xxy}-T_{yxx}+T_{yyy})],\\
T^{211} &= \frac{1}{4}[(1+\mathrm{i})(2T_{xxy}-T_{yxx}+T_{yyy})+(1-\mathrm{i})(T_{xxx}-T_{xyy}+2T_{yxy})],\\
T^{212} &= \frac{1}{4}[(1+\mathrm{i})(T_{xxx}+T_{xyy})+(1-\mathrm{i})(T_{yxx}+T_{yyy})]=T^{221},\\
T^{222} &= \frac{1}{4}[(1+\mathrm{i})(2T_{xxy}+T_{yxx}-T_{yyy})+(1-\mathrm{i})(-T_{xxx}+T_{xyy}+2T_{yxy})].
\end{aligned} \tag{24}$$

It is worth noting that in this case, for Eqs. (20), there are only three complex components to be specified to know $\mathbf{T}$: T^{111}, T^{112}, T^{122}. Unlike tensors of an even order, the introduction of symmetries do not render real some contravariant components.

4. Change of frame by a rotation

Let us now consider a change of frame by a rotation, of amplitude θ, that performs the passage from the frame $\{x, y\}$ to the frame $\{x', y'\}$. Following the original approach of Verchery, the rotation matrix $\mathbf{R}_1$, which makes the transformation for the contravariant components of a vector $\mathbf{X}^{\text{cont}}$ through the relation

$$\mathbf{X}'^{\text{cont}} = \mathbf{R}_1 \mathbf{X}^{\text{cont}}, \tag{25}$$

is

$$\mathbf{R}_1 = \begin{bmatrix} r & 0 \\ 0 & \bar{r} \end{bmatrix}, \tag{26}$$

where $r = \mathrm{e}^{-\mathrm{i}\theta}$. The rotation matrices for higher order tensors can be constructed by a recursive formula like that used for matrices $\mathbf{m}_j$, Eq. (15), so that for a third order tensor it is:

$$\mathbf{R}_3 = \begin{bmatrix} R_1^{11}\mathbf{R}_2 & R_1^{12}\mathbf{R}_2 \\ R_1^{21}\mathbf{R}_2 & R_1^{22}\mathbf{R}_2 \end{bmatrix} = \begin{bmatrix} r^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{r}^3 \end{bmatrix}. \tag{27}$$

Like any other rotation matrix $\mathbf{R}_j$, $\mathbf{R}_3$ is diagonal; actually, this is a consequence of the Verchery transformation, (8), but of course this is not the case in the usual matrix notation. In addition, and unlike the case of even order tensors, all the components of $\mathbf{R}_3$ are complex: this means that, being

$$\mathbf{T}'^{\text{cont}} = \mathbf{R}_3 \mathbf{T}^{\text{cont}}, \tag{28}$$

no one of the components of a third order tensor is invariant under a rotation. Finally, by using (18) and (28), one finds easily that, for what concerns the Cartesian components of $\mathbf{T}$, they are linked by the, well known, relation,

$$\mathbf{T}' = \mathbf{r}_3 \mathbf{T}, \tag{29}$$

with (for conciseness, $c = \cos\theta$, $s = \sin\theta$)

$$\mathbf{r}^3 = \mathbf{m}_3^{-1}\mathbf{R}_3\mathbf{m}_3 = \begin{bmatrix} c^3 & sc^2 & sc^2 & cs^2 & sc^2 & cs^2 & cs^2 & s^3 \\ -sc^2 & c^3 & -cs^2 & sc^2 & cs^2 & sc^2 & -s^3 & cs^2 \\ -sc^2 & -cs^2 & c^3 & sc^2 & -cs^2 & -s^3 & sc^2 & cs^2 \\ cs^2 & -sc^2 & -sc^2 & c^3 & s^3 & -cs^2 & -cs^2 & sc^2 \\ -sc^2 & -cs^2 & -cs^2 & -s^3 & c^3 & sc^2 & sc^2 & cs^2 \\ cs^2 & -sc^2 & s^3 & -cs^2 & -sc^2 & c^3 & -cs^2 & sc^2 \\ cs^2 & s^3 & -sc^2 & -cs^2 & -sc^2 & -cs^2 & c^3 & sc^2 \\ -s^3 & cs^2 & cs^2 & -sc^2 & cs^2 & -sc^2 & -sc^2 & c^3 \end{bmatrix}. \tag{30}$$

5. Tensor invariants under a rotation of a piezoelectricity-like tensor

Let **T** be a third order plane tensor (that is an element of $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$) with components satisfying conditions (23), like in the case of the piezoelectricity tensor; we want to find an expression for the invariants of the tensor under a frame rotation. The six independent contravariant components of **T** in the new frame $\{x', y'\}$ are, according to (27) and (28),

$$\begin{aligned}
T'^{111} &= r^3 T^{111},\\
T'^{112} &= r T^{112},\\
T'^{122} &= \bar{r} T^{122},\\
T'^{211} &= r T^{211},\\
T'^{221} &= \bar{r} T^{221},\\
T'^{222} &= \bar{r}^3 T^{222}.
\end{aligned} \tag{31}$$

As already remarked in the previous section, all the components of $\mathbf{R}_3$ are complex, so that no one of the contravariant components of **T** are invariant under a rotation. This implies that there are not linear combinations of the contravariant components of **T** that can be invariant under a frame rotation, i.e. **T** does not have linear invariants.

To find quadratic invariants, the following five quantities are introduced (Eqs. (20) are used here):

$$\begin{aligned}
&D_1 + iD_2 = T'^{112} T'^{122} = r\bar{r} T^{112} T^{122} = T^{112} T^{122}, \quad D_1, D_2 \in \mathbb{R},\\
&D_1 - iD_2 = T'^{211} T'^{221} = r\bar{r} T^{211} T^{221} = T^{211} T^{221},\\
&D_3 = T'^{112} T'^{221} = r\bar{r} T^{112} T^{221} = T^{112} T^{221} = T^{112} \bar{T}^{112}, \quad D_3 \in \mathbb{R},\\
&D_4 = T'^{122} T'^{211} = r\bar{r} T^{122} T^{211} = T^{122} T^{211} = T^{122} \bar{T}^{122}, \quad D_4 \in \mathbb{R},\\
&D_5 = T'^{111} T'^{222} = r^3\bar{r}^3 T^{111} T^{222} = T^{111} T^{222} = T^{111} \bar{T}^{111}, \quad D_5 \in \mathbb{R}.
\end{aligned} \tag{32}$$

It must be remarked that Eq. (32_b) is introduced considering that $\mathrm{Re}(T^{112}T^{122}) = \mathrm{Re}(T^{211}T^{221})$ and that $\mathrm{Im}(T^{112}T^{122}) = -\,\mathrm{Im}(T^{211}T^{221})$; these equalities are a consequence of Eqs. (20), that is of the fact, due to the transformation of Verchery, that a tensor component is equal to the complex conjugate of the component obtained exchanging indexes 1 and 2.

Considering the following syzygy relation:

$$D_1^2 + D_2^2 = (D_1 + iD_2)(D_1 - iD_2) = T^{112} T^{122} T^{211} T^{221} = T^{112} T^{122} \bar{T}^{122} \bar{T}^{112} = D_3 D_4, \tag{33}$$

the quadratic independent invariants of **T** are four: D_1, or D_2, D_3, D_4 and D_5.

Cubic invariants do not exist, because of the same reason for which there are no linear invariants. Invariants of the fourth order can be obtained considering the following quantities (once more, Eqs. (20) are used here):

$$\begin{aligned}
&Q_1 + iQ_2 = T'^{111}(T'^{122})^3 = r^3\bar{r}^3 T^{111}(T^{122})^3 = T^{111}(T^{122})^3, \quad Q_1, Q_2 \in \mathbb{R},\\
&Q_3 + iQ_4 = T'^{111}(T'^{221})^3 = r^3\bar{r}^3 T^{111}(T^{221})^3 = T^{111}(T^{221})^3,\\
&Q_1 - iQ_2 = T'^{222}(T'^{211})^3 = r^3\bar{r}^3 T^{222}(T^{211})^3 = T^{222}(T^{211})^3,\\
&Q_3 - iQ_4 = T'^{222}(T'^{112})^3 = r^3\bar{r}^3 T^{222}(T^{112})^3 = T^{222}(T^{112})^3,\\
&(D_1 + iD_2)^2 = (T'^{112})^2 (T'^{122})^2 = r^2\bar{r}^2 (T^{112})^2 (T^{122})^2 = (T^{112} T^{122})^2,\\
&(D_1 - iD_2)^2 = (T'^{221})^2 (T'^{211})^2 = r^2\bar{r}^2 (T^{221})^2 (T^{211})^2 = (T^{221} T^{211})^2,\\
&D_3^2 = (T'^{112})^2 (T'^{221})^2 = r^2\bar{r}^2 (T^{112})^2 (T^{221})^2 = (T^{112} \bar{T}^{112})^2,\\
&D_4^2 = (T'^{122})^2 (T'^{211})^2 = r^2\bar{r}^2 (T^{122})^2 (T^{211})^2 = (T^{122} \bar{T}^{122})^2
\end{aligned} \tag{34}$$

and the syzygy relations

$$
\begin{aligned}
&Q_1^2+Q_2^2=(Q_1+\mathrm{i}Q_2)(Q_1-\mathrm{i}Q_2)=T^{111}(T^{122})^3\bar{T}^{111}(\bar{T}^{122})^3=D_5D_4^3,\\
&(Q_1+\mathrm{i}Q_2)(Q_3-\mathrm{i}Q_4)=T^{111}(T^{122})^3\bar{T}^{111}(\bar{T}^{221})^3=T^{111}\bar{T}^{111}(T^{122}T^{112})^3=D_5(D_1+\mathrm{i}D_2)^3.
\end{aligned}
\tag{35}
$$

Eqs. (35_b) and (35_d) are introduced in the same way and for the same reason of Eq. (32_b). Eq. (35_b) gives the two relations (corresponding to the real and imaginary parts)

$$
\begin{aligned}
&Q_1Q_3+Q_2Q_4=D_5(D_1^3-3D_1D_2^2),\\
&Q_2Q_3-Q_1Q_4=D_5(3D_1^2D_2-D_2^3),
\end{aligned}
\tag{36}
$$

so that three real relations exist among the four real quantities Q_1, Q_2, Q_3 and Q_4, Eqs. (35_a) and (36). Hence, only one among the four invariants Q_1, Q_2, Q_3 and Q_4 is independent. Finally, there are five invariants under rotations, say D_1, D_3, D_4, D_5 and Q_1. It is worth noting that the relation

$$
Q_3^2+Q_4^2=(Q_3+\mathrm{i}Q_4)(Q_3-\mathrm{i}Q_4)=T^{111}\bar{T}^{111}(T^{112}\bar{T}^{112})^3=D_5D_3^3,
\tag{37}
$$

analogous to (35_a), is already comprised in Eqs. (36); in fact, from these equations one can get

$$
\begin{aligned}
&Q_3=\frac{D_5(Q_1D_1^3-3Q_1D_1D_2+3D_1^2D_2Q_2-D_2^3Q_2)}{Q_1^2+Q_2^2},\\
&Q_4=-\frac{D_5(3Q_1D_1^2D_2-Q_2D_1^3+3D_1D_2^2Q_2-D_2^3Q_1)}{Q_1^2+Q_2^2},
\end{aligned}
\tag{38}
$$

and if the syzygy relations are used in Eqs. (38), Eq. (37) is easily found.

The Cartesian expression of the five invariants D_1, D_3, D_4, D_5 and Q_1 can be easily found introducing Eqs. (24) into (32) and (34). Although straightforward, calculations are rather cumbersome; the results, given hereafter, show that to find the invariants directly by a Cartesian approach would be very difficult:

$$
\begin{aligned}
D_1={}&\frac{1}{8}(T_{xxx}^2+2T_{xxx}T_{yyx}+2T_{yxx}T_{xxy}-T_{yxx}^2+2T_{yyy}T_{xxy}+T_{yyy}^2-T_{xyy}^2+2T_{xyy}T_{yyx}),\\
D_3={}&\frac{1}{8}(T_{xxx}^2+2T_{xxx}T_{xyy}+T_{yxx}^2+2T_{yyy}T_{yxx}+T_{yyy}^2+T_{xyy}^2),\\
D_4={}&\frac{1}{8}(T_{xxx}^2-2T_{xxx}T_{xyy}+4T_{xxx}T_{yyx}-4T_{yxx}T_{xxy}+T_{yxx}^2-2T_{yxx}T_{yyy}+4T_{xxy}T_{yyy}+T_{yyy}^2+T_{xyy}^2\\
&-4T_{xyy}T_{yyx}+4T_{yyx}^2+4T_{xxy}^2),\\
D_5={}&\frac{1}{8}(T_{xxx}^2-2T_{xxx}T_{xyy}-4T_{xxx}T_{yyx}+4T_{yxx}T_{xxy}+T_{yxx}^2-2T_{yxx}T_{yyy}-4T_{xxy}T_{yyy}+T_{yyy}^2+T_{xyy}^2\\
&+4T_{xyy}T_{yyx}+4T_{yyx}^2+4T_{xxy}^2),\\
Q_1={}&\frac{1}{64}\Big[12(T_{yxx}T_{xyy}^2T_{yyy}+T_{xxx}T_{xyy}^2T_{yyx}+4T_{xxx}T_{xxy}^2T_{yyx}-T_{xxx}T_{yxxx}^2T_{yyx}+T_{xxx}T_{xyy}T_{yxx}^2\\
&-T_{xxx}T_{yyy}^2T_{yyx}+T_{xxx}T_{xyy}T_{yyy}^2+T_{xxy}T_{yxx}^2T_{yyy}-T_{xxy}T_{yxx}T_{yyy}^2+T_{xxy}T_{xyy}^2T_{yxx}\\
&-4T_{xxy}T_{yxx}T_{yyx}^2+4T_{xxy}T_{yyy}T_{yyx}^2+T_{xxy}T_{xxx}^2T_{yxx}-T_{xxy}T_{xxx}^2T_{yyy}+T_{yxx}T_{xxx}^2T_{yyy}^{-}4T_{xyy}T_{xxy}^2T_{yyx}\\
&-T_{xxx}^2T_{xyy}T_{yyx}+8T_{xxy}^2T_{yyx}^2-2T_{xxx}T_{xxy}T_{xyy}T_{yxx}+2T_{xxx}T_{xxy}T_{xyy}T_{yyy}-2T_{xxx}T_{yxx}T_{xyy}T_{yyy}\\
&+2T_{xxx}T_{yxx}T_{yyy}T_{yyx}-2T_{xyy}T_{yxx}T_{yyy}T_{yyx}+T_{xyy}T_{yxx}^2T_{yyx}+T_{xyy}T_{yyy}^2T_{yyx}-T_{xxy}T_{xyy}^2T_{yyy})\\
&+6(T_{yxx}^2T_{yyy}^2-T_{xyy}^2T_{yxx}^2-T_{xyy}^2T_{yyy}^2-T_{xxx}^2T_{yxx}^2-T_{xxx}^2T_{yyy}^2+T_{xxx}^2T_{xyy}^2)\\
&+4(T_{xxx}^3T_{yyx}-T_{xxx}^3T_{xyy}-T_{xxy}T_{yxx}^3+T_{xxy}T_{yyy}^3-4T_{xxy}^3T_{yyy}+4T_{xxy}^3T_{yxx}+4T_{xyy}T_{yyx}^3-T_{yxx}T_{yyy}^3\\
&-T_{yxx}^3T_{yyy}-4T_{xxy}^4-T_{xxx}T_{xyy}^3-4T_{xxx}T_{yyx}^3-4T_{yyx}^4-T_{xyy}^3T_{yyx})+T_{xyy}^4+T_{xxx}^4+T_{yyy}^4+T_{yxx}^4\Big].
\end{aligned}
\tag{39}
$$

6. Polar components for a piezoelectricity-like plane tensor

As already said in Section 3, for Eqs. (20) there are only three complex contravariant components to be specified to know a piezoelectricity-like tensor **T**: T^{111},T^{112},T^{122}. Then, it can be posed:

$$\begin{aligned}T^{111} &= -2A_1\mathrm{e}^{3\mathrm{i}\alpha_1},\\ T^{112} &= \mathrm{i}A_2\mathrm{e}^{\mathrm{i}\alpha_2},\\ T^{122} &= 2A_3\mathrm{e}^{-\mathrm{i}\alpha_3}.\end{aligned} \tag{40}$$

The other components can be obtained using relations (20). In Eqs. (40), the six polar components of **T** have been introduced: the three moduli A_1, A_2, A_3 and the three polar angles α_1, α_2, α_3. The Cartesian components of **T** can be easily expressed as functions of the above polar parameters simply using the inverse of Eq. (18), Eqs. (14_d) and (40):

$$\begin{aligned}T_{xxx} &= A_1\cos 3\alpha_1 + A_1\sin 3\alpha_1 - A_2\cos\alpha_2 - A_2\sin\alpha_2 + A_3\cos\alpha_3 - A_3\sin\alpha_3,\\ T_{xxy} &= -A_1\cos 3\alpha_1 + A_1\sin 3\alpha_1 + A_3\cos\alpha_3 + A_3\sin\alpha_3,\\ T_{xyy} &= -A_1\cos 3\alpha_1 - A_1\sin 3\alpha_1 - A_2\cos\alpha_2 - A_2\sin\alpha_2 - A_3\cos\alpha_3 + A_3\sin\alpha_3,\\ T_{yxx} &= -A_1\cos 3\alpha_1 + A_1\sin 3\alpha_1 + A_2\cos\alpha_2 - A_2\sin\alpha_2 - A_3\cos\alpha_3 - A_3\sin\alpha_3,\\ T_{yyx} &= -A_1\cos 3\alpha - A_1\sin 3\alpha_1 + A_3\cos\alpha_3 - A_3\sin\alpha_3,\\ T_{yyy} &= A_1\cos 3\alpha_1 - A_1\sin 3\alpha_1 + A_2\cos\alpha_2 - A_2\sin\alpha_2 + A_3\cos\alpha_3 + A_3\sin\alpha_3.\end{aligned} \tag{41}$$

The three complex equations reversals of Eqs. (41) can be easily obtained considering Eqs. (24) and (40):

$$\begin{aligned}8A_1\mathrm{e}^{3\mathrm{i}\alpha_1} &= (1+\mathrm{i})(T_{xxx} - T_{xyy} - 2T_{yyx}) + (1-\mathrm{i})(T_{yyy} - T_{yxx} - 2T_{xxy}),\\ 4A_2\mathrm{e}^{\mathrm{i}\alpha_2} &= -(1+\mathrm{i})(T_{xxx} + T_{xyy}) + (1-\mathrm{i})(T_{yxx} + T_{yyy}),\\ 8A_3\mathrm{e}^{-\mathrm{i}\alpha_3} &= (1+\mathrm{i})(T_{xxx} - T_{xyy} + 2T_{yyx}) + (1-\mathrm{i})(T_{yyy} - T_{yxx} + 2T_{xxy}).\end{aligned} \tag{42}$$

If now a frame rotation of amplitude θ is considered, after what said in Section 4 it will be

$$\begin{aligned}T'^{111} &= r^3T^{111} = -2A_1\mathrm{e}^{3\mathrm{i}(\alpha_1-\theta)},\\ T'^{112} &= rT^{112} = \mathrm{i}A_2\mathrm{e}^{\mathrm{i}(\alpha_2-\theta)},\\ T'^{122} &= \bar{r}T^{122} = 2A_3\mathrm{e}^{-\mathrm{i}(\alpha_3-\theta)},\end{aligned} \tag{43}$$

so that also the Cartesian components T'_{ijk} of **T** in the new frame $\{x', y'\}$ are simply obtained subtracting the rotation angle θ from the three polar angles in Eqs. (41): this is another well known advantage of the polar method.

7. Analysis of material symmetries

A question arises: are the invariants under a rotation also invariants for symmetry with respect to an axis turned of the angle α from the x-axis? To answer to this question, the original approach of Verchery is followed: such a symmetry can be represented by the transformation

$$\begin{aligned}z' &= s^2\bar{z},\\ \bar{z}' &= \bar{s}^2 z,\end{aligned} \tag{44}$$

with $s = \mathrm{e}^{\mathrm{i}\alpha}$. Applying again the transformation of Verchery, Eq. (8), it is

$$\begin{aligned}X'^1 &= -\mathrm{i}s^2X^2,\\ X'^2 &= \mathrm{i}\bar{s}^2X^1,\end{aligned} \tag{45}$$

and going through the same steps of the case of rotations, the following anti-diagonal transformation matrix is found

$$\mathbf{S}_3=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}s^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}s^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{i}s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{i}\bar{s}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i}s^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}\bar{s}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{i}\bar{s}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{i}\bar{s}^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{46}$$

Hence, for a piezoelectricity-like tensor it is

$$\begin{aligned} T'^{111} &= \mathrm{i}s^6 T^{222}, \\ T'^{112} &= -\mathrm{i}s^2 T^{221}, \\ T'^{122} &= \mathrm{i}\bar{s}^2 T^{211}, \\ T'^{211} &= -\mathrm{i}s^2 T^{122}, \\ T'^{221} &= \mathrm{i}\bar{s}^2 T^{112}, \\ T'^{222} &= -\mathrm{i}\bar{s}^6 T^{111}. \end{aligned} \tag{47}$$

Eqs. (47) show that there are not contravariant components of $\mathbf{T}$ that are invariant for a symmetry.

To study the effect of a material symmetry with respect to an axis rotated through the angle α from the x-axis, let us consider a rotation to the frame $\{x', y'\}$ whose bisector is just the symmetry axis. The rotation to pass from the frame $\{x, y\}$ to the frame $\{x', y'\}$ is then $\theta = \alpha - \frac{\pi}{4}$, so that $r = \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} = k\bar{s}$ and the transformation is, see Eq. (31),

$$\begin{aligned} T'^{111} &= -\bar{k}\bar{s}^3 T^{111}, \\ T'^{112} &= k\bar{s} T^{112}, \\ T'^{122} &= \bar{k}s T^{122}, \\ T'^{211} &= k\bar{s} T^{211}, \\ T'^{221} &= \bar{k}s T^{221}, \\ T'^{222} &= -ks^3 T^{222}. \end{aligned} \tag{48}$$

If now we consider the new frame $\{x'', y''\}$ such that $x'' = y'$ and $y'' = x'$, then

$$T'^{111} = T'^{222} = -ks^3 T^{222}, \tag{49}$$

but, for the symmetry and Eqs. (20), it is also

$$T''^{111} = T''^{222} = T'^{111} = -\bar{k}\bar{s}^3 T^{111} = -\bar{k}\bar{s}^3 \bar{T}^{222}, \tag{50}$$

so that, for Eq. (49),

$$T''^{111} = \bar{T}''^{111}. \tag{51}$$

This last equation implies that the quantity $k\ s^3 T^{222} = k\ \mathrm{e}^{3\mathrm{i}\alpha} T^{222}$ is real, as well as its complex conjugate $\bar{k}\bar{s}^3 T^{111} = \bar{k}\mathrm{e}^{-3\mathrm{i}\alpha} T^{111}$. In the same way, it can be shown that the quantities $k\bar{s} T^{112} = k\mathrm{e}^{-\mathrm{i}\alpha} T^{112}$ and $\bar{k}s T^{122} = \bar{k}\mathrm{e}^{\mathrm{i}\alpha} T^{122}$ are real as well.

If now these results are introduced in the expressions of the invariants $D_1 + \mathrm{i}D_2$, $Q_1 + \mathrm{i}Q_2$ and $Q_3 + \mathrm{i}Q_4$, Eqs. (32_a), ($34_{a,b}$), then it is immediately seen that these invariants must be real, i.e.

$$D_2 = Q_2 = Q_4 = 0. \tag{52}$$

To examine more deeply the effects of the symmetry, let us consider the conditions (52) in the syzygy relations, Eqs. (33), (35_a) and (37):

$$\begin{aligned} &D_1^2 = D_3 D_4, \\ &Q_1^2 = D_5 D_4^3, \\ &Q_3^2 = D_5 D_3^3. \end{aligned} \tag{53}$$

The third condition (53) is redundant. The three conditions (53) can also be condensed in

$$\left(\frac{Q_1}{Q_3}\right)^2 = \left(\frac{D_4}{D_3}\right)^3 = \left(\frac{D_1}{D_3}\right)^6; \tag{54}$$

finally, only three invariants are independent, say D_1, D_4 and Q_1. By inserting Eqs. (40) in Eqs. ($32_{c,d,e}$) and (34_a) it follows that

$$\begin{aligned} &D_1^2 = 4A_2^2 A_3^2 \sin^2(\alpha_2 - \alpha_3), \\ &D_3 D_4 = 4A_2^2 A_3^2, \\ &Q_1^2 = 256 A_1^2 A_3^6 \cos^2 3(\alpha_1 - \alpha_3), \\ &D_5 D_4^3 = 256 A_1^2 A_3^6, \end{aligned} \tag{55}$$

so that, due to Eqs. ($53_{a,b}$), the symmetry conditions are

$$\begin{cases} \cos^2 3(\alpha_1 - \alpha_3) = 1 \quad \Longleftrightarrow \quad \alpha_1 - \alpha_3 = k_1 \frac{\pi}{3}, \\ \sin^2(\alpha_2 - \alpha_3) = 1 \quad \Longleftrightarrow \quad \alpha_2 - \alpha_3 = \frac{\pi}{2} + k_2 \pi, \end{cases} \tag{56}$$

or

$$\begin{cases} A_1 = 0, \\ A_2 = 0, \end{cases} \tag{57}$$

or also

$$A_3 = 0. \tag{58}$$

The symmetry conditions (57) implies that

$$D_1 = D_3 = D_5 = Q_1 = 0, \quad D_4 \neq 0, \tag{59}$$

while condition (58) implies

$$D_1 = D_4 = Q_1 = 0, \quad D_3 \neq 0, \ \ D_5 \neq 0. \tag{60}$$

Two other, redundant, conditions of symmetry are $A_1 = A_3$=0, which gives identically $T_{xxy} = T_{yyx}$=0, and $A_2 = A_3$=0.

The direction α of symmetry is linked to the value of the polar angles α_1 and α_2; this circumstance can be proved by considering that, as remarked above, the quantities $\bar{k}e^{-3i\alpha}T^{111}$, $\bar{k}e^{i\alpha}T^{122}$ and $ke^{-i\alpha}T^{112}$ are real and accounting for (40). Trivial algebra implies the conditions

$$\begin{aligned} &\tan\alpha = \frac{\sin\alpha_2 - \cos\alpha_2}{\sin\alpha_2 + \cos\alpha_2} = \frac{\cos\alpha_3 + \sin\alpha_3}{\cos\alpha_3 - \sin\alpha_3}, \\ &\tan 3\alpha = \frac{\sin 3\alpha_1 - \cos 3\alpha_1}{\sin 3\alpha_1 + \cos 3\alpha_1}. \end{aligned} \tag{61}$$

Finally, Eqs. (55) show also that A_1, A_2, A_3 and the angular differences $\alpha_1 - \alpha_3$ and $\alpha_2 - \alpha_3$ are tensor invariants. Indeed, only one of the three polar angles must be chosen, and this corresponds to the choice of the reference frame.

8. An energetic interpretation of the polar components of a piezoelectricity tensor

An energetic interpretation of the polar components of a constitutive tensor for two dimensional piezoelectric bodies can be done. To this aim, consider the electric enthalpy H given by

$$H(\mathbf{S},\mathbf{E}) = \tfrac{1}{2}\mathbb{C}_{ijkl}S_{ij}S_{kl} - \tfrac{1}{2}\varepsilon_{ij}E_iE_j - e_{ijk}E_iS_{jk}, \tag{62}$$

where $\mathbb{C}$ is the elasticity tensor at null electric field, $\mathbf{S}$ the strain tensor, $\mathbf{E}$ the electric field, $\boldsymbol{\varepsilon}$ the dielectric permittivity tensor at null strain field and $\mathbf{e}$ the piezoelectricity tensor, which couples the deformation and the electric field (in the bi-dimensional case, the different tensors can be, if the case, replaced by the respective reduced tensors); we are interested here with the term

$$H_p(\mathbf{S},\mathbf{E}) = -e_{ijk}E_iS_{jk}. \tag{63}$$

Let us represent $\mathbf{e}$, $\mathbf{S}$ and $\mathbf{E}$ by the polar method (see for instance Verchery, 1979, or Vannucci, 2005):

$$\mathbf{e} = \left\{\begin{array}{llllllll} e_{xxx} = & A_1\cos 3\alpha_1 & +A_1\sin 3\alpha_1 & -A_2\cos\alpha_2 & -A_2\sin\alpha_2 & +A_3\cos\alpha_3 & -A_3\sin\alpha_3 \\ e_{xxy} = & -A_1\cos 3\alpha_1 & +A_1\sin 3\alpha_1 & & & +A_3\cos\alpha_3 & +A_3\sin\alpha_3 \\ e_{xyy} = & -A_1\cos 3\alpha_1 & -A_1\sin 3\alpha_1 & -A_2\cos\alpha_2 & -A_2\sin\alpha_2 & -A_3\cos\alpha_3 & +A_3\sin\alpha_3 \\ e_{yxx} = & -A_1\cos 3\alpha_1 & +A_1\sin 3\alpha_1 & +A_2\cos\alpha_2 & -A_2\sin\alpha_2 & -A_3\cos\alpha_3 & -A_3\sin\alpha_3 \\ e_{yyx} = & -A_1\cos 3\alpha & -A_1\sin 3\alpha_1 & & & +A_3\cos\alpha_3 & -A_3\sin\alpha_3 \\ e_{yyy} = & A_1\cos 3\alpha_1 & -A_1\sin 3\alpha_1 & +A_2\cos\alpha_2 & -A_2\sin\alpha_2 & +A_3\cos\alpha_3 & +A_3\sin\alpha_3 \end{array}\right\}, \tag{64}$$

$$\mathbf{S} = \left\{\begin{array}{lll} S_{xx} = & T+ & R\cos 2\Phi \\ S_{xy} = & & R\sin 2\Phi \\ S_{yy} = & T- & R\cos 2\Phi \end{array}\right\}, \tag{65}$$

$$\mathbf{E} = \left\{\begin{array}{l} E_x = L\cos\lambda \\ E_y = L\sin\lambda \end{array}\right\}. \tag{66}$$

In Eqs. (65), T and R are the polar invariants of $\mathbf{S}$, while Φ is the polar angle, frame dependent, while in Eqs. (66) L is the Euclidean norm of $\mathbf{E}$ and λ the angle that the vector $\mathbf{E}$ forms with the x-axis.

Using Eqs. (63)–(66) the following expression of H_p as function of the polar components of $\mathbf{e}$, $\mathbf{S}$ and $\mathbf{E}$ is easily found:

$$\begin{aligned} H_p = 2L\{&A_1R[\sin(2\Phi+\lambda-3\alpha_1) - \cos(2\Phi+\lambda-3\alpha_1)] + A_2T[\cos(\lambda-\alpha_2) - \sin(\lambda-\alpha_2)] \\ &- A_3R[\sin(2\Phi-\lambda-\alpha_3) + \cos(2\Phi-\lambda-\alpha_3)]\}. \end{aligned} \tag{67}$$

So, for what said in Section 1 about T and R, A_1 and A_3 are responsible of the part of H_p which depends upon the deviator of $\mathbf{S}$, whilst A_2 for the spherical part. If, for example, the deformation is spherical, R=0 and if the vector $\mathbf{E}$ is in the direction α_2, then

$$H_p = -2LA_2T, \tag{68}$$

i.e., H_p is proportional, through $2A_2$, to the spherical part of $\mathbf{S}$ and to the norm of $\mathbf{E}$.

9. Conclusions and perspectives

The polar method proposed by Verchery has been applied to the search of the invariants of a third order piezoelectricity-like plane tensor. Five invariants have been found and the polar components of the tensor have been introduced. The link between the invariants, as well as the polar components, and a material symmetry has been discussed. Finally, an energetic interpretation of the polar components has been given for the case of the piezoelectricity tensor. The polar description of plane anisotropic elasticity has already proved to be effective, especially in the description of the elastic properties and in the optimal design of laminates; in the same way, the polar description of the piezoelectricity tensor can be usefully used in the design of piezoelectric anisotropic plates with special plane actuators, already existing in some industrial applications; this perspective has mainly motivated this research and we are investigating now in this direction.

References

Green, A., Zerna, W., 1954. Theoretical Elasticity. Clarendon Press, Oxford, UK.

Milton, G.W., 2002. The Theory of Composites. Cambridge University Press, UK.

Vannucci, P., 2001. On bending-tension coupling of laminates. Journal of Elasticity 64, 13–28.

Vannucci, P., 2002. A special planar orthotropic material. Journal of Elasticity 67, 81–96.

Vannucci, P., 2002. Thèse pour l'habilitation à diriger les recherches. HDR thesis. University of Burgundy (in French).

Vannucci, P., 2005. Plane anisotropy by the polar method. In: Meccanica, Vol. 40. Springer, pp. 437–454. doi:10.1007/s11012-005-2132-z.

Vannucci, P., Verchery, G., 2001. Stiffness design of laminates using the polar method. International Journal of Solids and Structures 38, 9281–9294.

Verchery, G., 1979. Les invariants des tenseurs d'ordre 4 du type de l'élasticité. Proc. of Colloque Euromech 115, Villard-de-Lans. Published in 1982 by Editions du CNRS, Paris, 93–104.